

Fixed-energy harmonic functions

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Abstract

We study the map from conductances to edge energies for harmonic functions on finite graphs with Dirichlet boundary conditions. We prove that for any compatible acyclic orientation and choice of energies there is a unique choice of conductances such that the associated harmonic function realizes those orientations and energies. We call the associated function *enharmonic*. For rational energies and boundary data the Galois group of \mathbb{Q}^{tr} (the totally real algebraic numbers) over \mathbb{Q} permutes the enharmonic functions, acting on the set of compatible acyclic orientations.

For planar graphs there is an enharmonic conjugate function; together these form the real and imaginary parts of a “fixed energy” analytic function. In the planar scaling limit for \mathbb{Z}^2 (and the fixed south/west orientation), these functions satisfy a nonlinear analog of the Cauchy-Riemann equations, namely

$$\begin{aligned}u_x v_y &= 1 \\u_y v_x &= -1.\end{aligned}$$

We give an analog of the Riemann mapping theorem for these functions, as well as a variational approach to finding solutions in both the discrete and continuous settings.

1 Overview

1.1 The Dirichlet problem

The classical Dirichlet problem is to find a harmonic function on a domain that takes specified values on the boundary. It is the mathematical abstraction of physical equi-

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librium problems arising in electromagnetism, fluid flow, gravitation, and other areas. Solutions to the Dirichlet problem describe physical phenomena from the vibrations of a drum to the dissipation of heat. Dirichlet himself was investigating the stability of the solar system [7]. In the century and a half since Dirichlet's work, there have been countless investigations into various versions, modifications, generalizations, and special cases of the Dirichlet problem.

We study the discrete Dirichlet problem on a graph, and in particular we are interested in the connection between the *edge conductances* and the *edge energies* of the resulting solution. In the classical problem one fixes the conductances, and then minimizing the Dirichlet energy gives a unique solution to the Dirichlet problem, namely a harmonic function on the vertices called the *voltage*. The edge energies are then obtained from this solution. Thus there is a map, soon called Ψ , from edge conductances to edge energies.

Here we reverse the problem and fix the desired energies of the harmonic function, determining which choices of conductances lead to these energies. We show (Theorem 9) that in suitable coordinates Ψ has a surprisingly simple Jacobian, which makes analysis of the point preimages very clean. This set of preimages is parameterized by a certain set of acyclic orientations of the graph (Theorems 1 and 3), and one can determine the Dirichlet solutions through a simple equation, the *enharmonic Laplace equation* (Theorem 2). Remarkably, the cardinality of the solution set is independent of the boundary values, as long as these are distinct (Corollary 4).

We begin in Section 1.2 with notation and an overview of our main results showing existence of solutions and giving variational descriptions of these solutions. In Sections 1.3, 1.4, 1.5, 1.6 we give consequences of our theorems that highlight connections between the Dirichlet problem on a graph and other areas of mathematics, such as number theory (Theorem 6), discrete analytic functions (Theorem 11), and rectangle tilings (Corollaries 7 and 8). Some proofs are given; longer ones are deferred to later sections.

Several explicit examples of the main theorems are given in Section 2.

In Section 3 we study the Jacobian of Ψ . Sections 4, 5, 6 contain proofs. We pose a few questions in Section 7.

1.2 Main results

Let $\mathcal{G} = (V, E)$ be a connected graph and $B \subset V$ a subset of ≥ 2 vertices, which we refer to as *boundary vertices*. We assume that each edge of \mathcal{G} lies on a simple path from a vertex of B to another vertex of B . Let $c_e > 0$ be a *conductance* for each edge e . It will sometimes be convenient to refer to edges by their endpoints, e.g., we also write c_{xy} for the same conductance if e is the edge with endpoints x, y .

Given a function $u : B \rightarrow \mathbb{R}$ the classical Dirichlet problem is to find a function f

equal to u on B and which minimizes the *Dirichlet energy*

$$\mathcal{E}(f) = \sum_{e=xy} c_e(f(x) - f(y))^2,$$

where the sum is over all edges of \mathcal{G} . There is a unique minimizing function h ; it is the unique function harmonic at each vertex of $V \setminus B$, that is satisfying, for $x \notin B$,

$$0 = \Delta h(x) := \sum_{y \sim x} c_e(h(x) - h(y))$$

where the sum is over neighbors y of x . We define the *energy* of h on an edge $e = xy$ to be $c_e(h(x) - h(y))^2$, and

$$\Psi = \Psi_u : (0, \infty)^E \rightarrow [0, \infty)^E \quad (1)$$

to be the map from the set of edge conductances to the set of energies of the associated harmonic function h . See an example in (6) below.

Our main theorem relates the conductances and edge energies of a harmonic function. (See the definition of compatible orientation below.)

Theorem 1 (Existence). *Let the graph \mathcal{G} , boundary B , and boundary values $u : B \rightarrow \mathbb{R}$ be fixed. For each compatible orientation σ of \mathcal{G} and tuple $\mathcal{E} \in (0, \infty)^E$ there is a unique choice of positive conductances $\{c_e\}$ on edges so that the associated harmonic function h has $\text{sgn}(dh) = \sigma$ and energies \mathcal{E} .*

Let \mathcal{F}_u be the set of real-valued functions on \mathcal{G} which have no interior extrema, that is, no local maxima or local minima except on B . For any choice of positive conductances, a harmonic function with boundary values u on B necessarily lies in \mathcal{F}_u , by the maximum principle for harmonic functions.

Suppose that $f \in \mathcal{F}_u$ takes distinct values at the two endpoints of each edge of \mathcal{G} . Then f induces an orientation of the edges of \mathcal{G} , from larger values to smaller values. This orientation has the following three obvious properties:

1. it is acyclic, that is, there are no oriented cycles,
2. it has no interior sinks or sources (as these would correspond to interior extrema of f),
3. there are no oriented paths from lower boundary values to higher (or equal) boundary values.

Given u , we call an orientation satisfying the above properties *compatible (with u)*, and let Σ_u be the set of orientations compatible with u . It is not hard to see that for each $\sigma \in \Sigma_u$ there is a function $h \in \mathcal{F}_u$ inducing σ , that is, with $dh(e) := h(x) - h(y)$

nonzero on every edge $e = xy$, and having sign $\text{sgn}(dh(e)) = \sigma(e)$ on each edge. The set $\mathcal{F}_u(\sigma)$ of functions $h \in \mathcal{F}_u$ inducing the orientation σ is therefore an *open polytope* $\mathcal{F}_u(\sigma)$, since it is defined by (strict) linear inequalities.

The most basic setting of Theorem 1 above is the case where the boundary B consists in just two vertices. Let $B = \{v_0, v_1\}$ and $u(v_0) = 0, u(v_1) = 1$. Then Σ_u consists of acyclic orientations which have a unique sink, at v_0 , and a unique source, at v_1 . These are called *bipolar orientations*. Their cardinality is the *beta invariant* of the graph $\tilde{\mathcal{G}}$, in which an edge is added to \mathcal{G} from v_0 to v_1 if there is no such edge already present. The beta invariant, also known as the *chromatic invariant*, is the derivative at 1 of the chromatic polynomial of $\tilde{\mathcal{G}}$, or equivalently the coefficient of x (and of y) in the Tutte polynomial $T(x, y)$, see [4].

By Theorem 1 above, if we fix a graph \mathcal{G} with boundary B , a function $u : B \rightarrow \mathbb{R}$, and on each edge e an energy $\mathcal{E}_e > 0$, the set of functions in \mathcal{F}_u with energies $\{\mathcal{E}_e\}$ is in bijection with the set Σ_u of compatible orientations of \mathcal{G} . We call these functions *enharmonic* (short for “energy-harmonic”) with respect to the energies $\{\mathcal{E}_e\}$. Note that a function that is enharmonic with respect to certain energies is also harmonic with respect to certain conductances, and vice versa; the difference in terminology indicates only which data were used to define the function.

Theorem 2 (Characterization). *Let the graph \mathcal{G} , boundary B , and boundary values $u : B \rightarrow \mathbb{R}$ be fixed. Enharmonic functions with energies $\mathcal{E} \in (0, \infty)^E$ are precisely the local maxima on \mathcal{F}_u of the functional*

$$M(h) = \prod_{e=xy \in E} |h(x) - h(y)|^{\mathcal{E}_e}. \quad (2)$$

The enharmonic function h with energies \mathcal{E} and satisfying $dh = \sigma$ is the unique maximizer of $M(h)$ on the polytope $\mathcal{F}_u(\sigma)$. An enharmonic function is one which satisfies the (discrete) enharmonic Laplace equation $Lh(x) = 0$ at every interior vertex x , where L is the nonlinear operator

$$Lh(x) = \sum_{y \sim x} \frac{\mathcal{E}_{xy}}{h(x) - h(y)} \quad (3)$$

and the sum is taken over all vertices y adjacent to x .

In practice, for given energies $\{\mathcal{E}_e\}$, determining the function h and the conductances $\{c_e\}$ is most easily done via the equation (3). We prove Theorem 2 in Section 5.

1.3 The degree of Ψ

The map Ψ_u of (1) from conductances to energies is a rational map (i.e. its coordinate functions are ratios of polynomial functions of the conductances). To see this, note first

that to obtain the harmonic function f as a function of the conductances and boundary data, we have to invert the Laplacian which has entries which are linear polynomials in the conductances; the energies are then obtained from $\mathcal{E}_e = c_e(f(x) - f(y))^2$. Thus Ψ_u can be extended to a rational map

$$\Psi_u : \mathbb{C}^E \rightarrow \mathbb{C}^E.$$

The inverse map Ψ_u^{-1} is algebraic. Remarkably, for positive real energies all inverse images are positive real:

Theorem 3. *The rational map Ψ_u is of degree $|\Sigma_u|$ over \mathbb{C} .*

By this we mean that there is an open dense set of \mathbb{C}^E such that the cardinality of the preimage of any point in this set is $|\Sigma_u|$. See Section 4 for a proof. Since Theorem 1 provides $|\Sigma_u|$ preimages of any point with positive real coordinates, such a preimage must consist only of points of \mathbb{C}^E with positive real coordinates:

$$\Psi_u^{-1}((0, \infty)^E) \subset (0, \infty)^E.$$

An immediate consequence is

Corollary 4. *The number of compatible orientations $|\Sigma_u|$ does not depend on the values of u , as long as u takes distinct values at distinct boundary vertices.*

Proof. The degree of a rational family of rational maps is constant for almost every parameter value. Thus for almost every choice of values of u , the degree of Ψ_u is constant. As u varies among a (full-dimensional) set of functions taking distinct values, the set of compatible orientations does not change. So the set of u 's where the degree of Ψ_u is smaller is a subset of the set u 's where at least two boundary values are the same. \square

1.4 Galois action

Suppose the given energies and boundary values lie in \mathbb{Q} , with positive energies. Then the enharmonic equation

$$\sum_{y \sim x} \frac{\mathcal{E}_{xy}}{h(x) - h(y)} = 0 \tag{4}$$

supplemented with the boundary equations $h|_B = u$ which define the enharmonic functions, form a rational system of equations defined over \mathbb{Q} . Thus the absolute Galois group permutes the solutions. Theorem 3 shows that the values of f generate a totally real number field. Thus we have

Corollary 5. *If the values of u and the coordinates of $\mathcal{E} \in (0, \infty)^E$ are rational then the Galois group of \mathbb{Q}^{tr} (the totally real algebraic numbers) over \mathbb{Q} permutes the enharmonic functions of Theorem 1. Equivalently, $\text{Gal}(\mathbb{Q}^{\text{tr}}/\mathbb{Q})$ acts on Σ_u .*

We conjecture that for 3-connected graphs, the action is transitive¹:

Conjecture 1. *If G is 3-connected the above Galois action is transitive for generic² rational energies and boundary data.*

Considering all acyclic orientations $\cup \Sigma_u$ as \mathcal{G} and u vary, this is reminiscent of *dessins d'enfants*, a family of combinatorial objects on which the absolute Galois group $\text{Gal}(\mathbb{Q}/\mathbb{Q})$ acts. In the current setup it is the Galois group of the totally real algebraic numbers which acts. We prove (see Section 6) that the action is faithful, i.e., that every totally real number field arises in this way:

Theorem 6. *For every totally real number field K there is a tuple $(\mathcal{G}, B, u, \{\mathcal{E}_e\})$ consisting of a graph \mathcal{G} , boundary B , rational boundary values u and rational edge energies $\mathcal{E}_e > 0$ such that K is the number field generated by the values of one of the associated enharmonic functions, and all enharmonic functions generate isomorphic number fields.*

We don't know if Theorem 6 holds when the size of the boundary is constrained, for example when the boundary consists in only two points. We show this however for quadratic fields in Section 2.2.

1.5 Discrete analytic functions

A *circular planar network* [5] is a finite embedded planar graph \mathcal{G} with conductances $c : E \rightarrow \mathbb{R}_{>0}$ and boundary B consisting of a subset of vertices on the outer face. We define the *dual network* \mathcal{G}^* as the usual planar dual of the graph obtained from \mathcal{G} by adding disjoint rays from each boundary vertex of \mathcal{G} to infinity. Thus \mathcal{G}^* has one vertex in every bounded face of \mathcal{G} and one vertex in the exterior face between every pair of “consecutive” boundary vertices of \mathcal{G} ; there is a dual edge connecting dual vertices for every primal edge separating the associated faces and there is a dual edge for each of the infinite rays. Given a harmonic function f on \mathcal{G} with boundary values u on B , a function g on \mathcal{G}^* is said to be a *harmonic conjugate* if for any primal edge $e = xy$ we have $f(y) - f(x) = c_e(g(t) - g(s))$ where s, t are the faces adjacent to edge xy , with t being on the right when the edge is traversed from x to y . These equations are

¹A graph with boundary is 3-connected if, upon adding edges connecting every pair of boundary vertices, it cannot be disconnected by removing two vertices.

²By generic we mean on a Zariski open set.

the *discrete Cauchy-Riemann equations*. They imply in fact that f is harmonic on \mathcal{G} , and g is harmonic on the dual graph with reciprocal conductances on the dual edges. In this setting such a pair f, g is also said to form a *discrete analytic function*.

For a circular planar network $(\mathcal{G}, B, u, \{\mathcal{E}_e\})$ with enharmonic function f , an *enharmonic conjugate* g is a function g on \mathcal{G}^* satisfying

$$(g(t) - g(s))(f(y) - f(x)) = \mathcal{E}_{xy},$$

where s, t are the faces adjacent to edge xy , with t being on the right when the edge is traversed from x to y . Enharmonicity of f implies enharmonicity of g on the dual graph with the same energies. Together f and g are harmonic conjugates for the associated conductances. We call the pair (f, g) the *fixed-energy discrete analytic function* associated to the boundary data. These functions are used as x - and y -coordinates in the rectangle tilings discussed in the next section.

It is natural to consider scaling limits of enharmonic functions, that is, continuous limits of enharmonic functions on a sequence of graphs approximating (in an appropriate sense) a region in \mathbb{R}^n . These scaling limits depend strongly on both the underlying graph structure and choice of orientation. We consider here only the simplest nontrivial case where the approximating graph is a scaled subgraph of \mathbb{Z}^2 , with edges oriented south and west. We prove in Theorem 11 below that the natural scaling limit of an enharmonic function f and its enharmonic conjugate g for $\mathcal{G} = \epsilon\mathbb{Z}^2$ satisfy a nonlinear analog of the Cauchy-Riemann equation:

$$\begin{aligned} f_x g_y &= -1 \\ f_y g_x &= 1. \end{aligned} \tag{5}$$

Moreover f, g both satisfy the “continuous enharmonic Laplace equation”

$$\frac{f_{xx}}{f_x^2} + \frac{f_{yy}}{f_y^2} = 0.$$

We prove (see Section 5.3) an analog of the Riemann mapping theorem in this context, namely that from any region in an appropriate class there is a “fixed-energy analytic” map to a rectangle.

In [1] a different continuous analog of the fixed-energy problem is considered.

1.6 Tilings and networks

In 1940, Brooks, Smith, Stone, and Tutte [3], building on work of Dehn [6], gave a correspondence between the Dirichlet problem for circular planar networks (with two boundary vertices) and tilings of a rectangle by rectangles; they called this the *Smith*

diagram of the network. See Figure 2 below for an example. Namely, given such a tiling, one can orient the rectangle so it sits on the x -axis, and then form a network from the tiling according to the following rules. Place a vertex at height y for each maximal horizontal segment in the tiling at height y ; then, for each rectangular tile add an edge to the network connecting the vertices corresponding to the top and bottom of the tile. On this edge place a conductance which is the aspect ratio (width over height) of the tile. The result is a circular planar graph which can be interpreted as a resistor network.

If we now imagine the top and bottom vertices connected by a battery generating a voltage differential equal to the height of the rectangle, then the voltage function on the vertices (i.e., the y -values) is a harmonic function that solves the Dirichlet problem on the network with boundary conditions enforced at the top and bottom vertex. The rectangle widths are the current flows $c_e dh$, and the areas are the edge energies $c_e (dh)^2$.

The BSST correspondence is invertible: beginning with a planar network with a specified source and sink on the outer face and a specified total voltage drop, one can solve for the harmonic function and the currents in the wires. The resulting network can then be represented as a rectangle tiling by reversing the process described above.

A corollary of Theorem 1 applies to Smith diagrams. This was originally proved in a different manner in [12] and a number of times since, see [8].

Corollary 7. *Given a rectangle tiling of a rectangle there is an isotopic tiling in which the rectangles have prescribed areas.*

Here by *isotopic* we mean one is obtained from the other by displacing continuously the walls, floors and ceilings of rectangles in the first to form the second, without degenerating any rectangles along the way (we also need to add a proviso that at a point where four rectangles meet only one of the two possible perturbations is allowed; either the horizontal segments to its left and right can move independently up and down, or the vertical segments above and below can move independently left and right, but not both.) See Section 2 for some examples.

Such maps are also called *rectangular cartograms* [9] and *floorplans* [2] and are used e.g. to make geographic maps where the area represents the measure of some quantity associated to each country, such as population.

Another corollary of Theorems 1 and 3 applies to the tilability of polygons by rectangles of rational areas. Such a tiling arises from a circular planar network with a boundary vertex for each horizontal segment of the boundary polygon. Again the Dirichlet boundary values are the y -coordinates of the horizontal boundary segments.

Corollary 8. *If a polygon P can be tiled with rectangles of rational areas, then the horizontal edge lengths are in a totally real extension field of $\mathbb{Q}(v_1, \dots, v_k)$, the field generated by the vertical edge lengths.*

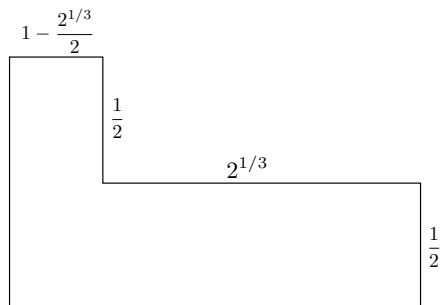


Figure 1: A polygon with area 1 which is not tilable with rational-area rectangles.

For example the unit-area polygon of Figure 1 cannot be tiled with rectangles of rational areas, because $2^{1/3}$ is not a totally real algebraic number.

Proof. Assume some vertex of P is at $(0,0)$. As discussed above, the y -coordinates of the horizontal edges of P correspond to the boundary values u of the enharmonic function. The y -coordinates of the interior horizontal edges are the interior values of the enharmonic function, and therefore in a totally real extension of $\mathbb{Q}(v_1, \dots, v_k)$. The x -coordinates of all vertical edges are sums of currents $\mathcal{E}_{xy}/(f(x) - f(y))$, which are therefore in the same extension field. \square

2 Examples

We illustrate our theorems and techniques with a few examples.

2.1 A small graph

Consider the graph \mathcal{G} shown in Figure 2, with conductances a, b, c, d, e as labeled. With $u(v_1) = 1$ and $u(v_0) = 0$, we have $x, y \in [0, 1]$, and thus $\mathcal{F}_u \cong [0, 1]^2$. Solving for h gives $h(x) = \frac{1}{Z}(ab + ac + bc + ae)$ and $h(y) = \frac{1}{Z}(ab + ac + bc + bd)$, where $Z = ab + ac + ae + bc + bd + cd + ce + de$ is the determinant of the Laplacian. The

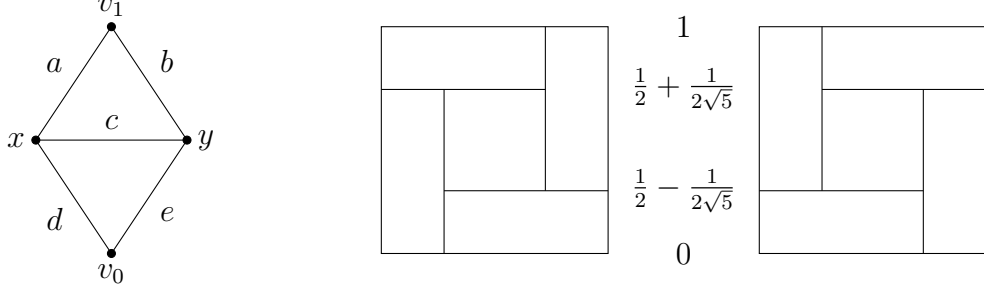


Figure 2: A graph with $|\Sigma| = 2$. With $u(v_0) = 0, u(v_1) = 1$ and all energies equal, the underlying number field is $\mathbb{Q}(\sqrt{5})$.

energies are then

$$\Psi_u(a, b, c, d, e) = \frac{1}{Z^2} \left(a(bd + cd + ce + de)^2, b(ac + cd + ce + de)^2, c(ae - bd)^2, d(ab + ac + bc + ae)^2, e(ab + ac + bc + bd)^2 \right) \quad (6)$$

To find solutions with (for instance) all energies equal to 1, it is much easier to solve the enharmonic equation $Lh = 0$ than it is to set each coordinate above equal to 1 and solve the system directly. The enharmonic equation(s) (see (4)) are

$$\begin{aligned} 0 &= \frac{\mathcal{E}_a}{h(x) - h(v_1)} + \frac{\mathcal{E}_c}{h(x) - h(y)} + \frac{\mathcal{E}_d}{h(x) - h(v_0)} = \frac{1}{h(x) - 1} + \frac{1}{h(x) - h(y)} + \frac{1}{h(x)} \\ 0 &= \frac{\mathcal{E}_b}{h(y) - h(v_1)} + \frac{\mathcal{E}_c}{h(y) - h(x)} + \frac{\mathcal{E}_e}{h(y) - h(v_0)} = \frac{1}{h(y) - 1} + \frac{1}{h(y) - h(x)} + \frac{1}{h(y)} \end{aligned}$$

and have the solutions $h(x) = \frac{1}{2} \pm \frac{\sqrt{5}}{10}$ and $h(y) = \frac{1}{2} \mp \frac{\sqrt{5}}{10}$, from which one can quickly produce the conductances (for example, $a = e = \frac{1}{2}(15 \pm 5\sqrt{5})$). The values of h are roots of the polynomial $5z^2 - 5z + 1$. There are two solutions because current can flow either way along the edge xy . That is, $|\Sigma| = 2$. The two polytopes in $\mathcal{F}_u = [0, 1]^2$ on which $M(h)$ is nonzero are the triangles above and below the diagonal.

The equal-area tilings associated to the two solutions are shown in Figure 2; the y -coordinates of the horizontal edges are $h(x)$ and $h(y)$. The tilings are not isotopic.

2.2 Quadratic number fields

Keeping the same graph \mathcal{G} as in the previous example, if we instead look for solutions with arbitrary energies $\mathcal{E}_a, \mathcal{E}_b, \mathcal{E}_c, \mathcal{E}_d, \mathcal{E}_e$, we find that the conductances generate the

number field $\mathbb{Q}(\sqrt{\delta})$, where

$$\delta = (\mathcal{E}_a\mathcal{E}_e - \mathcal{E}_b\mathcal{E}_d + \mathcal{E}_cS)^2 + 4\mathcal{E}_b\mathcal{E}_c\mathcal{E}_dS.$$

Here S is the sum of the five energies. Of course for positive real energies, $\delta > 0$ as all roots are real. It turns out that we can get any real quadratic number field this way. To see this, we first specify $\mathcal{E}_a = \mathcal{E}_b = \mathcal{E}_c = 1$ so that the discriminant reduces to $\delta = 4\mathcal{E}_d^2 + 4\mathcal{E}_d\mathcal{E}_e + 4\mathcal{E}_e^2 + 12\mathcal{E}_d + 12\mathcal{E}_e + 9$. To obtain the number field $\mathbb{Q}(\sqrt{D})$ for a positive squarefree integer D , it suffices to find positive rational numbers $\mathcal{E}_d, \mathcal{E}_e$ such that $\delta = Dr^2$ for some rational r . Choose a rational s so that $D' = Ds^2 \in (1/3, 4/9)$. Then substituting $\mathcal{E}_d = \frac{1}{6D'-2}, \mathcal{E}_e = \frac{4-9D'}{6D'-2}$ (both positive) gives discriminant $\delta = \frac{9}{(3D'-1)^2}D'$ which is a rational square times D .

2.3 Jacobi polynomials

One way to generalize the previous graph is the following.³ Let $\mathcal{G} = K_{n+2}$ be the complete graph on $n+2$ vertices v_0, \dots, v_{n+1} , minus the edge v_0v_1 . Fix boundary $B = \{v_0, v_1\}$ and boundary values $u(v_0) = 0$ and $u(v_1) = 1$. Put energy a on all edges from v_0 to v_i , $i \in [2, n+1]$ and energy b on edges from v_1 to v_i , $i \in [2, n+1]$, and energy 2 on the remaining edges. Then by symmetry we can order the values $h(i)$ for $i \in [2, n+1]$ in increasing order. We have

$$M(h) = \prod_{i \in [2, n+1]} h(i)^a (1 - h(i))^b \prod_{1 < i < j < n+1} (h(j) - h(i))^2.$$

In [11, Theorem 6.7.1] it is shown that the values $h(i)$ maximizing $M(h)$ are the roots of the Jacobi orthogonal polynomial $P_{a-1, b-1}(z)$, scaled to the interval $[0, 1]$.

2.4 A medium graph

For another example, consider the network shown in Figure 3. Again let $u(v_1) = 1$ and $u(v_0) = 0$. The edges are labeled 1 through 12, and (by coincidence) also $|\Sigma_u| = 12$. We have solved the enharmonic equation (see (4)) for the twelve enharmonic functions that produce all energies equal to 1. From these functions we computed the conductances of the networks, which determine the shapes of the rectangles in the corresponding Smith diagrams. In Figure 4 we have scaled the Smith diagrams to be (unit) squares, so each rectangle in each rectangulation has area $1/12$. Any rectangulation of a rectangle with the same underlying network (Figure 3) is isotopic to one of these twelve.

³Note that $n = 2$ is the only member of this family that is planar.

These calculations are too complicated to do by hand; for instance, according to Mathematica, the width of the rectangle labeled “1” in each picture is a root of the polynomial

$$\begin{aligned} p(z) = & 1270080000000z^{12} - 5554584000000z^{11} + 10776143400000z^{10} \\ & - 12235337185000z^9 + 9034493949125z^8 - 4560532680000z^7 \\ & + 1610724560815z^6 - 400501165895z^5 + 69535433439z^4 \\ & - 8223166134z^3 + 629396649z^2 - 28041714z + 551124. \end{aligned}$$

All other edge lengths in all tilings in Figure 4 lie in the splitting field of p , which is contained in \mathbb{R} .

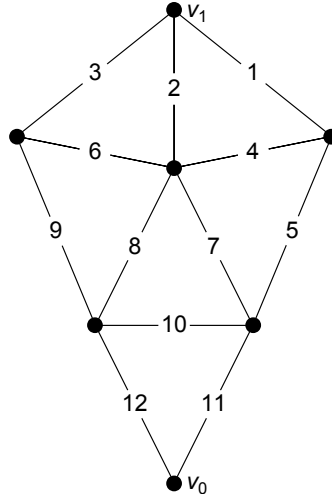


Figure 3: A planar network with two boundary vertices v_0, v_1 .

2.5 A large graph

The variational method makes it very easy to find (numerically) enharmonic functions on large graphs, see for example Figure 5. In fact this figure illustrates a certain scaling limit convergence theorem, Theorem 11 below.

3 The Jacobian

The work in this paper began with the observation that the Jacobian of Ψ has a surprisingly simple description, which we give here. Our proof of Theorem 1 uses the nonvanishing of this Jacobian.

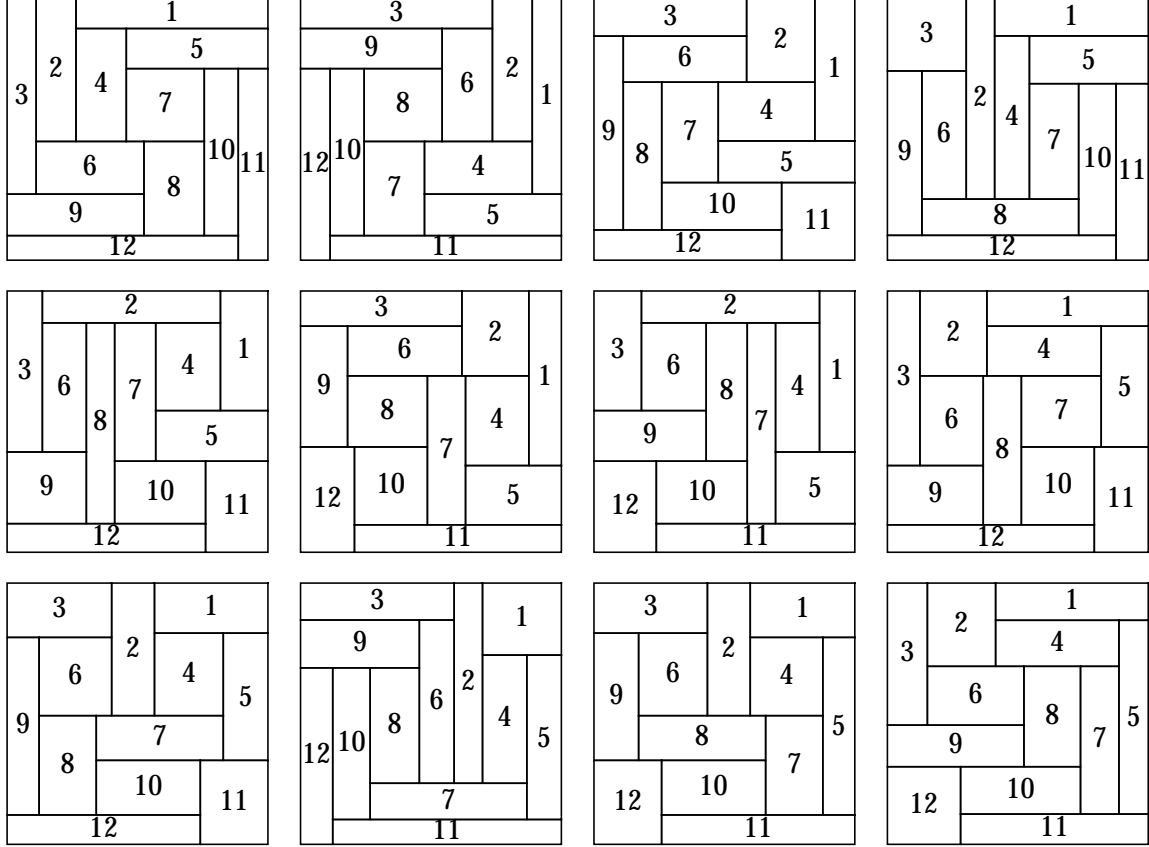


Figure 4: Smith diagrams for the planar network of Figure 3, with energies 1. Here $|\Sigma_u| = 12$, and the Galois group acts transitively: the underlying number field for each tiling is a degree-12 totally real extension field of \mathbb{Q} . Each tiling corresponds to a different bipolar orientation of \mathcal{G} , so these tilings are not isotopic.

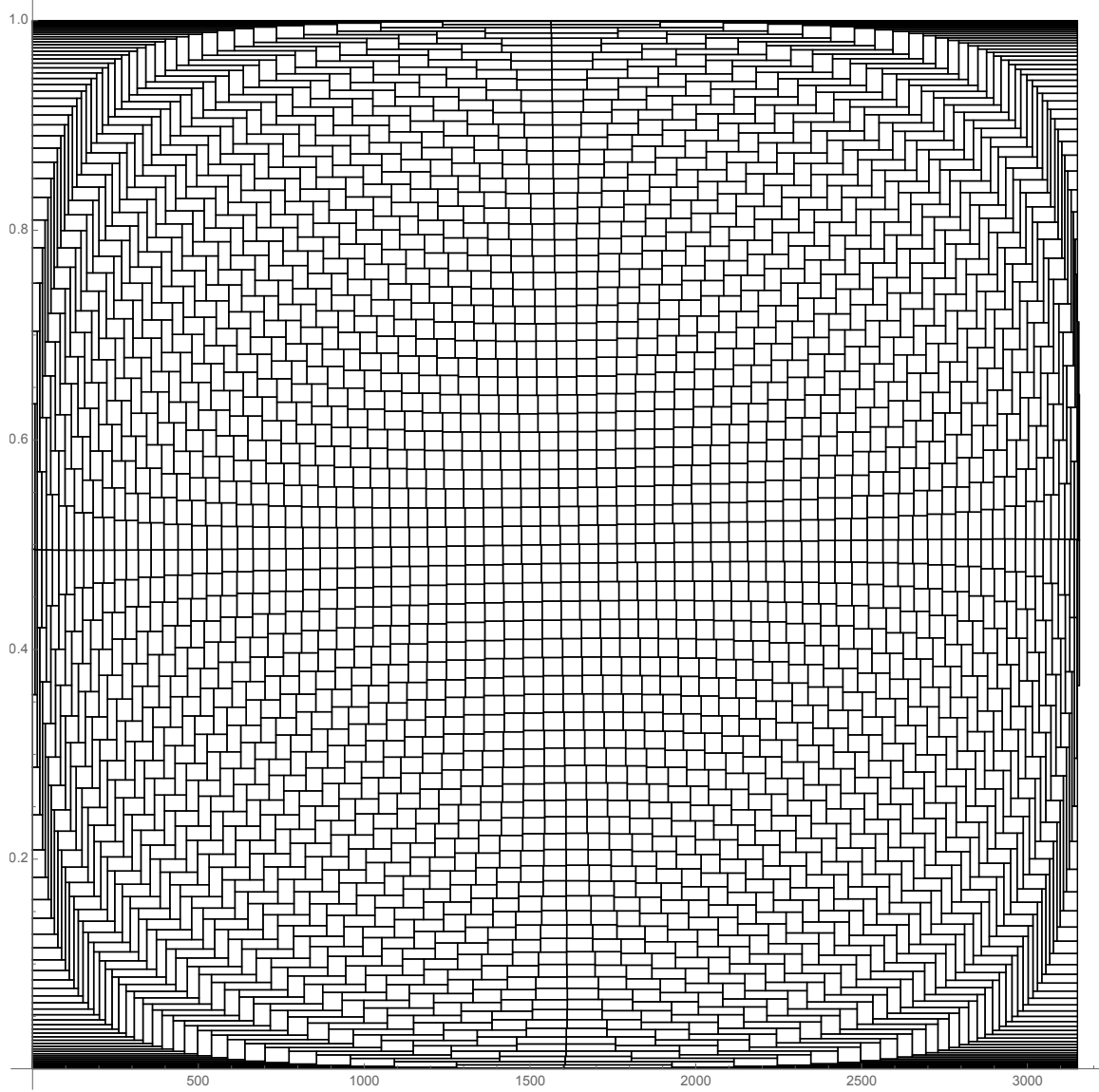


Figure 5: Smith diagram for the 40×40 square grid $[0, 40]^2 \cap \mathbb{Z}^2$, with edges oriented south and west, with all energies 1 and boundary values $u(0, 0) = 0$, $u(40, 40) = 1$. The apparent regularity here can be quantified: see Theorem 11.

First some definitions. On a connected graph $\mathcal{G} = (V, E)$ with boundary vertices B , a 1-form is a function ω on oriented edges which is antisymmetric under changing orientation: $\omega(-e) = -\omega(e)$ where $-e$ represents the edge e with reversed orientation. Let Γ^1 be the space of 1-forms. Let Γ^0 be the space of functions on V with zero boundary values.

We define the map $d : \Gamma^0 \rightarrow \Gamma^1$ by $df(e) = f(y) - f(x)$ when $e = \vec{xy}$ is directed from x to y . A 1-form is a *coboundary* if it is of the form df for some $f \in \Gamma^0$. Let $W_{cob} \subset \Gamma^1$ be the space of coboundaries. It has a natural basis consisting of the set of coboundaries $\{d\mathbb{1}_v\}_{v \in V \setminus B}$ where $\mathbb{1}_v$ is the function which is 1 at v and zero elsewhere.

The orthogonal complement (for the standard inner product on \mathbb{R}^E) of W_{cob} is the space W_{cyc} of (*relative*) *cycles*, or *flows*. These are 1-forms in the kernel of the boundary operator $d^* : \Gamma^1 \rightarrow \mathbb{R}^V / \mathbb{R}^B \cong \Gamma^0$, the transpose of d . The mapping d^* from 1-forms to functions is

$$d^*\omega(x) = \sum_{y \sim x} \omega(\vec{xy})$$

where the sum is over vertices y neighboring x ; in other terminology $d^*\omega$ is the *divergence* of ω . Cycles (or flows) are 1-forms which are divergence-free on the non-boundary vertices.

Note that W_{cob} has dimension $|V| - |B|$, the number of internal vertices, and W_{cyc} has dimension $|E| - |V| + |B|$.

The differential of the map Ψ_u has a surprisingly simple form when considered as a map from the logarithms of the conductances to the logarithms of the energies. We assume the boundary values u are fixed. If we fix an element $\sigma \in \Sigma_u$, we can identify \mathbb{R}^E with the space of 1-forms.

Let $P_{cyc} : \Gamma^1 \rightarrow \Gamma^1$ be the orthogonal projection onto W_{cyc} , and $P_{cob} = I - P_{cyc}$ the projection onto W_{cob} .

Theorem 9. *When $\Psi_u(\{c\})$ is nonzero on every edge, let J_{\log} be the differential of the map $\log \circ \Psi_u \circ \exp : \mathbb{R}^E \rightarrow \mathbb{R}^E$. Then $J_{\log} = P_{cyc} - P_{cob}$.*

Corollary 10.

$$\det D\Psi = (-1)^{|V|-|B|} \prod_e (h(x) - h(y))^2.$$

In particular on $(0, \infty)^E$, Ψ fails to be locally injective precisely when some edge has energy 0.

Proof of Corollary 10. By the Theorem (and the chain rule)

$$\det D\Psi = (-1)^{\dim(W_{cob})} \frac{\prod_e \mathcal{E}_e}{\prod_e c_e}.$$

Substitute $\mathcal{E}_e = c_e(h(x) - h(y))^2$ for edges $e = xy$. □

Proof of Theorem 9. Since $\mathbb{R}^E = W_{cob} \oplus W_{cyc}$, we study the action of J_{\log} on W_{cob} and W_{cyc} .

Given a conductance function c , let h be the associated harmonic function, and $\omega = c \cdot dh$ the associated current flow. We assume ω is nonzero on each edge, which holds for almost all choices of c .

Let $v \in V \setminus B$. We define a 1-parameter family of conductances $c_e(t)$, $t \in [0, 1]$ as follows. Let $h_1(v)$ be any real number close enough to $h(v)$ so that, if we redefine $h(v)$ to be $h_1(v)$ then the sign of dh does not change. Fix h at all vertices but v and let

$$h_t(v) = h(v) + t(h_1(v) - h(v)).$$

We fix ω to its initial value, and define the conductance at time t by $c_e(t) = \omega(e)/dh_t(e)$. Then h_t is harmonic for conductances $c_e(t)$. The associated energies are $\mathcal{E}_e(t) = \omega(e)dh_t(e)$ and thus $\frac{\partial}{\partial t} \log \mathcal{E}_e(t) = -\frac{\partial}{\partial t} \log c_e(t)$. Evaluating at $t = 0$, this vector is a multiple of $d\mathbb{1}_v$:

$$-\frac{\partial}{\partial t} \log c_e(t) \Big|_{t=0} = \frac{\partial}{\partial t} \log dh_t(e) \Big|_{t=0} = \frac{h_1(v) - h(v)}{h(v)} d\mathbb{1}_v.$$

Thus $d\mathbb{1}_v$ is an eigenvector of J_{\log} with eigenvalue -1 . As v varies over $V \setminus B$ the coboundaries $d\mathbb{1}_v$ span the coboundary space W_{cob} , so J_{\log} is minus the identity on W_{cob} .

Now let γ be a cycle in G . Let $\delta > 0$ be small enough so that $\omega_1 = \omega + \delta\gamma$ has the same sign as ω . We now fix h but vary ω according to $\omega_t(e) = \omega(e) + t(\omega_1(e) - \omega(e))$. Define the conductance at time t by $c_e(t) = \omega_t(e)/dh(e)$. We then have $\frac{\partial}{\partial t} \log \mathcal{E}_e = \frac{\partial}{\partial t} \log c_e(t)$. Evaluating at $t = 0$, this vector is a multiple of γ :

$$\frac{\partial}{\partial t} \log c_e(t) \Big|_{t=0} = \frac{\partial}{\partial t} \log \omega_t(e) \Big|_{t=0} = \frac{\omega_1(e) - \omega(e)}{\omega(e)} \gamma.$$

Thus J_{\log} is the identity on W_{cyc} . □

It is worth pointing out the interpretation of this proof in terms of tilings. If G is planar, the proof starts with a rectangle tiling with combinatorial structure described by G , and then studies the effect of isotopy on the diagram. Moving a horizontal segment corresponds to changing the value of h at a single point, which is an eigenvalue -1 of J_{\log} : the relative change of the conductances (widths over heights) move in the opposite direction as the relative change of areas (widths times heights). Moving a vertical segment corresponds to the eigenvalue $+1$: the relative change of the conductances move in the same direction as the relative change of areas.

4 Proofs of Theorems 1 and 3

We assume u is fixed and takes distinct values at distinct boundary vertices, and we omit it from the notation.

Proof of Theorem 1. We first prove that the map is surjective onto Σ , that is, every $\sigma \in \Sigma$ is realized by some choice of conductances. Given $\sigma \in \Sigma$, let f be any function in $\mathcal{F}(\sigma)$. Such a function exists because σ is acyclic: for example one can define f from a linear extension of σ , thinking of σ as a partial order. There is a choice of conductances for which this f is harmonic: to see this take any flow ω with orientation σ and divergence 0 on $V \setminus B$, and define the conductance c_e on edge $e = xy$ by $c_e = \frac{\omega(e)}{f(x) - f(y)}$. As a concrete example of such a flow ω , let P be the set of all possible oriented paths from a boundary vertex to a boundary vertex, with orientation compatible with σ , and let the value $\omega(e)$ be the probability that e is on a uniformly chosen path in P . By our condition on \mathcal{G} that each edge lies on a simple path from B to B , we see that ω is nonzero. This completes the proof that the map is surjective onto Σ .

Now consider the map $\Psi: (0, \infty)^E \rightarrow [0, \infty)^E$ sending a tuple of conductances to the corresponding tuple of energies of the associated harmonic function. Corollary 10 shows that the Jacobian of Ψ is nonzero when the energies are nonzero.

We claim that Ψ is proper: the preimage of a compact set is compact. In other words, Ψ maps the boundary to the boundary. Take a sequence of conductance functions leaving every compact set of $(0, \infty)^E$. By taking a subsequence, we can assume it converges to a boundary point of $[0, \infty]^E$. We need to show that along this sequence some edge has energy tending to 0 or ∞ .

Let S_0 be the set of edges whose conductances are going to zero, and S_∞ those whose conductances are going to ∞ . If S_∞ does not connect any two boundary vertices, then there is a limiting network obtained by contracting all edges in S_∞ . The resulting network has conductances bounded above, which implies a finite current flow. The energies on the contracted edges and any edges of zero conductance have gone to zero.

If S_∞ contains a connection between two boundary points (with different values of u , by genericity), the current flowing between these points will diverge. This implies that the energy (which can be written as current flow times dh) will diverge along some edges of this connection.

Thus Ψ is proper.

The preimage of $(0, \infty)^E$ in the space of conductances is a union of open sets U_σ where dh has constant sign σ . At a finite boundary point of any such U_σ , some energy goes to zero (since zero current implies zero energy). Hence the boundary of U_σ maps into the boundary of the space of energies. Thus Ψ on any of these open sets U_σ is proper and a local homeomorphism (a covering map), and therefore surjective from

this set to the space of energies. To get uniqueness it remains to show that U_σ is connected.

In fact we show that U_σ has the structure of (the interior of) a polytope. As in the first paragraph of the proof, a point in U_σ is determined by a flow compatible with σ (and of divergence zero on interior vertices), and an independent choice of function f in the open polytope $\mathcal{F}_u(\sigma)$. The set of current flows ω compatible with σ is the subset of $(0, \infty)^E$ defined by the linear equations $\text{div}(\omega) = 0$. It is the interior of a convex polytope (possibly infinite but with finitely many facets). \square

Proof of Theorem 3. We have constructed $|\Sigma|$ real preimages of each point in $(0, \infty)^E$. It remains to show that there are no non-real preimages. Let $\{c_e\} \in \Psi^{-1}(\{\mathcal{E}_e\})$ be a preimage of a set of positive real energies. Let h be the corresponding solution to the Dirichlet problem with conductances $\{c_e\}$. Then for all non-boundary vertices x ,

$$\sum_{y \sim x} c_e (h(x) - h(y)) = 0$$

and so

$$\sum_{y \sim x} \frac{\mathcal{E}_{xy}}{h(x) - h(y)} = 0.$$

We can rewrite this as

$$\sum_{y \sim x} \frac{\mathcal{E}_{xy}(\overline{h(x)} - \overline{h(y)})}{|h(x) - h(y)|^2} = 0,$$

or

$$\left(\sum_{y \sim x} \frac{\mathcal{E}_{xy}}{|h(x) - h(y)|^2} \right) \overline{h(x)} = \sum_{y \sim x} \frac{\mathcal{E}_{xy}}{|h(x) - h(y)|^2} \overline{h(y)}.$$

This represents $\overline{h(x)}$ as a convex combination of its neighboring values $\overline{h(y)}$, and thus $h(x)$ lies in the convex hull of the neighboring values $h(y)$. Since the boundary values of h are real, each $h(x)$ must be real as well. Finally $c_e = \mathcal{E}_e / (h(x) - h(y))^2$ so each c_e is real and positive. \square

5 Enharmonic functions: a variational method

5.1 Proof of Theorem 2

Proof of Theorem 2. Fix a compatible orientation σ . Recall the open polytope $\mathcal{F}_u(\sigma) \subset \mathcal{F}_u$ of functions inducing orientation σ . Restricted to $\mathcal{F}_u(\sigma)$, the function $\log M(h)$ is concave, since it is a sum of concave functions

$$\mathcal{E}(e) \log |h(x) - h(y)| = \mathcal{E}(e) \log(\sigma(e)(h(x) - h(y))).$$

The function $\log M(h)$ tends to $-\infty$ on the boundary of $\mathcal{F}_u(\sigma)$ (when $h(x) = h(y)$ for some edge $e = xy$). As a consequence $\log M(h)$, being analytic, is strictly concave on $\mathcal{F}_u(\sigma)$. It thus has a unique maximizer on the interior of $\mathcal{F}_u(\sigma)$, for each σ . The maximizer is determined by setting the derivative with respect to $h(x)$ equal to zero for each vertex x , that is

$$\sum_{y \sim x} \frac{\mathcal{E}(e)}{h(x) - h(y)} = 0. \quad (7)$$

Given a solution to (7), set $c_e = \mathcal{E}_e / (h(x) - h(y))^2$, then (7) becomes

$$\sum_{y \sim x} c_e (h(x) - h(y)) = 0, \quad (8)$$

that is, h is harmonic for the conductances c_e and has energies \mathcal{E}_e . □

5.2 Scaling limits

Let $D \subset \mathbb{R}^2$ be a Jordan domain with piecewise smooth boundary and let $u : \partial D \rightarrow \mathbb{R}$ be continuous. We assume u is *feasible* in the sense that the space $\Omega(D, u)$ of smooth functions $f : D \rightarrow \mathbb{R}$ with boundary values $f|_{\partial D} = u$ and satisfying $f_x, f_y > 0$ is nonempty⁴. Fix such a function $f_0 \in \Omega(D, u)$.

For sufficiently small $\epsilon > 0$ let $D_\epsilon = \epsilon \mathbb{Z}^2 \cap D$ be the part of the graph $\epsilon \mathbb{Z}^2$ contained in D . Let σ be the south and west orientation of the edges of D_ϵ . Let ∂D_ϵ be the set of boundary vertices of D_ϵ , that is, vertices with a nearest neighbor in $\epsilon \mathbb{Z}^2$ not in D_ϵ . Let $u_\epsilon : \partial D_\epsilon \rightarrow \mathbb{R}$ be the restriction of f_0 to ∂D_ϵ .

Let f_ϵ be the unique enharmonic function on D_ϵ with energy ϵ^2 per edge, with boundary values u_ϵ and with df_ϵ having orientation σ .

Theorem 11. *Under the above hypotheses, as $\epsilon \rightarrow 0$ the function f_ϵ converges to a continuous function on D , real analytic in the interior of D , and satisfying the enharmonic Laplace equation*

$$\frac{f_{xx}}{f_x^2} + \frac{f_{yy}}{f_y^2} = 0. \quad (9)$$

There is a unique (up to an additive constant) conjugate function g , also real analytic and continuous up to the boundary, satisfying (5).

⁴The issue of feasibility is not completely straightforward. Such a function f necessarily has level contours having tangents pointing in the NW and SE quadrants. Conversely, given a foliation \mathcal{F} of D by SE-directed leaves, in which the value of u on the endpoints of each leaf are the same, there is a function f whose level lines are the leaves of \mathcal{F} .

We call f satisfying (9) an *enharmonic function*. A function g satisfying (5) is an *enharmonic conjugate*.

Proof. Assuming $f \in C^2$ satisfies (9), the 1-form $\omega = -\frac{dx}{f_y} + \frac{dy}{f_x}$ is closed. Let $g(p) = \int_{p_0}^p \omega$ for some fixed $p_0 \in D$. Then f and g satisfy the “fixed-energy” Cauchy Riemann equations (5). The enharmonicity of g follows from (5) as well.

Let $F \in \Omega(D, u)$ be the (unique) function maximizing the concave functional

$$\mathcal{M}(F) = \int_D \log(F_x F_y) dx dy. \quad (10)$$

Existence and uniqueness of F both follow from strict concavity of \mathcal{M} (existence follows from the fact that the space of monotone functions with boundary values u is compact, and \mathcal{M} is upper semicontinuous; uniqueness follows from the fact that $\mathcal{M}(\frac{1}{2}(F_1 + F_2)) > \mathcal{M}(F_1) + \mathcal{M}(F_2)$ unless $F_1 = F_2$. Analyticity follows from analyticity of the function \log , see [10][Theorem 1.10.4 (vi)].

We will show that $f_\epsilon \rightarrow F$.

The Euler-Lagrange equation for a critical point of $\mathcal{M}(F)$ is the enharmonic equation

$$\frac{F_{xx}}{F_x^2} + \frac{F_{yy}}{F_y^2} = 0.$$

Let M_ϵ be the functional (2) for the graph D_ϵ with energies ϵ^2 . Let f_ϵ be its maximizer. After triangulating the faces of D_ϵ by adding diagonals, we can extend f_ϵ to a piecewise linear function \tilde{f}_ϵ on D (except near the boundary). Then $\log M_\epsilon(f_\epsilon)$ is a Riemann sum for $\mathcal{M}(\tilde{f}_\epsilon)$, so

$$\log M_\epsilon(f_\epsilon) = \mathcal{M}(\tilde{f}_\epsilon) + o(1).$$

If we restrict F to D_ϵ we likewise find by analyticity of F that

$$\log M_\epsilon(F) = \mathcal{M}(F) + o(1).$$

Consequently

$$\mathcal{M}(F) = \log M_\epsilon(F) + o(1) \leq \log M_\epsilon(f_\epsilon) + o(1) = \mathcal{M}(\tilde{f}_\epsilon) + o(1) \leq \mathcal{M}(F) + o(1).$$

Therefore all these quantities are within $o(1)$ of each other. By upper semicontinuity of \mathcal{M} , we find that f_ϵ converges to F .

We prove the second statement of the theorem in the coming section.

5.3 Riemann mapping to a rectangle

Let $D \subset \mathbb{R}^2$ be a piecewise smooth domain of the following type. There are four points a, b, c, d on the boundary of D in counter-clockwise order, so that the boundary of D between a and b is monotone in the NE direction (that is, is monotone in the $+x, +y$ directions), and similarly between b and c , c and d , d and a respectively the boundary is monotone in the NW, SW, and SE directions. Let u be the function which is 1 on ∂D between b and c , and zero between d and a . Let D_ϵ be as before the subgraph of $\epsilon\mathbb{Z}^2$ with vertices in D . Let $a_\epsilon, \dots, d_\epsilon$ be boundary vertices so that the boundary of D_ϵ is monotone NE, NW, SW, SE between a_ϵ and b_ϵ , b_ϵ and c_ϵ , c_ϵ and d_ϵ , and d_ϵ and a_ϵ . Let D'_ϵ be the dual graph of D_ϵ , with a vertex for every bounded face of D_ϵ and, instead of one exterior vertex, four exterior vertices $v_{ab}, v_{bc}, v_{cd}, v_{da}$ outside of the corresponding boundary edges of D_ϵ . Let σ be the south/west orientation of edges of D_ϵ .

Let $f \in \Omega(D, u)$ be the unique function on D with boundary values 1 on bc and 0 on da , free on the other boundaries, and maximizing $\mathcal{M}(f)$; this differs from the previous case because u is only fixed on part of the boundary, but again existence and uniqueness follow from concavity and analyticity from arguments of [10]. Then f will satisfy (9) on D , with fixed boundary values 0 on da and 1 on bc respectively and along the free boundary $ab \cup cd$, f will satisfy the condition that $\frac{dx}{f_y} - \frac{dy}{f_x} = 0$ (this is in fact the Euler-Lagrange condition along the free boundary).

As before the function f_ϵ on D_ϵ maximizing M will lie within $o(1)$ of f . Using edge energies ϵ^2 , let g_ϵ be the enharmonic conjugate of f_ϵ , defined on the dual graph D'_ϵ and with value zero on the dual vertex v_{ab} . Let $R_\epsilon = g_\epsilon(v_{cd})$ be the value at the opposite dual vertex. Since the sum of the energies is the area of the image under the rectangle tiling, we have

$$R_\epsilon = \epsilon^2(\# \text{ edges of } D_\epsilon) + o(1) = 2\text{Area}(D) + o(1).$$

The above convergence argument applied to g_ϵ shows that g_ϵ converges to g , the maximizer of \mathcal{M} with boundary values $A = 2\text{Area}(D)$ and 0 on cd and ab respectively.

It remains to show that f and g are enharmonic conjugates. The enharmonic conjugate f^* of f is the enharmonic function (well defined up to an additive constant) which is constant along ab and along cd , with free boundary values along bc and da . These are precisely the boundary conditions satisfied by the function g , except that the value of the constant along cd might be different, so we have that $f^* = Bg$ for some constant B . The map $(x, y) \mapsto (f^*, f)$ is area preserving except for a factor 2 (the Jacobian is identically 2, by (5)) so the image of D has area A , which means the boundary value of f^* along bc must also be A . Thus $f^* = g$.

This proves that (f_ϵ, g_ϵ) converges to (f, g) . In the limit we have a real-analytic mapping (g, f) from D to the rectangle $[0, A] \times [0, 1]$. \square

6 Realizing number fields

Proof of Theorem 6. Let \mathcal{G} be a $d + 1$ -star graph with vertices $\{v, v_0, \dots, v_d\}$, with v being the central vertex. Take boundary B consisting of all vertices except v . Let e_i be the energy on edge vv_i and take rational boundary values a_i at v_i , with $a_0 < \dots < a_d$. The enharmonic equation for the value x at v is

$$\sum_{i=0}^d \frac{e_i}{x - a_i} = 0. \quad (11)$$

This equation has exactly d roots, and these are real and interlaced with the a_i , that is, there is one in each interval (a_i, a_{i+1}) (this follows from Theorem 1 above or simply from the fact that the roots are roots of the derivative of $\prod (x - a_i)^{e_i}$).

Conversely let $p(x)$ be a polynomial with rational coefficients of degree d with all real roots, interlaced with the a_i as above. Then we claim that there is a unique choice of rational e_i so that the enharmonic equation (11) has exactly the same roots.

The map from the tuple of energies $\{e_i\}$ to the tuple of coefficients $\{b_i\}$ of the polynomial $p(x) = b_0 + b_1x + \dots + b_dx^d$ is a rational linear map from \mathbb{R}^{d+1} to itself. Specifically its matrix consists of elementary functions in the a_i (after removing a_{i-1} in those for the i th column) illustrated for the case $d = 3$:

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} -a_1a_2a_3 & -a_0a_2a_3 & -a_0a_1a_3 & -a_0a_1a_2 \\ a_1a_2 + a_3a_2 + a_1a_3 & a_0a_2 + a_3a_2 + a_0a_3 & a_0a_1 + a_3a_1 + a_0a_3 & a_0a_1 + a_2a_1 + a_0a_2 \\ -a_1 - a_2 - a_3 & -a_0 - a_2 - a_3 & -a_0 - a_1 - a_3 & -a_0 - a_1 - a_2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

The determinant of this matrix is a polynomial of degree $\binom{d}{2}$ and has factors $a_i - a_j$ for each $i < j$, so is the Vandermonde $\pm \prod_{i < j} (a_i - a_j)$ and is therefore nonzero. Thus the map from energies to coefficients b_i is invertible over \mathbb{Q} . This completes the proof of the claim.

Now let K be a totally real number field and $x \in K$ a primitive element (i.e. one for which $K = \mathbb{Q}[x]$). Let $p(X)$ be its minimal polynomial. Let r_1, \dots, r_d be the roots of $p(X)$, which are real. Replacing x with $x + m$ for a large enough integer m we can assume that all $r_i > 0$. Let a_1, \dots, a_{d+1} be positive rationals interlacing them. Now do the above construction with parameters a_i . \square

Note that as a simple corollary to this proof, by letting the e_i be arbitrary positive rationals, modulo a global scale, we can parameterize the polynomials with roots interlaced with the a_i as a simplex

$$\{(e_0, \dots, e_d) \mid \sum e_i = 1\}.$$

7 Other questions

1. Can one use the volume-preserving character of the map $\log \circ \Psi \circ \exp$ to compute the cardinality of Σ ? This is NP hard in general but integrating the Jacobian over projective space one can approximate the degree of Ψ and therefore $|\Sigma|$.
2. The minimal polynomials for geometric quantities (such as values of the harmonic function) seem to have coefficients which are (fixed-sign) integer-coefficient polynomials in the edge energies. Can one give a combinatorial description of these polynomials?
3. What can be said about solutions to the enharmonic equation in the case when the boundary values u are complex? The number of solutions is the same, but there is no longer the same interpretation in terms of orientations.

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